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The effect of weak inertia on flow through a porous medium

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Using the theory of homogenization we examine the correction to Darcy's law due to weak convective inertia of the pore fluid. General formulae are derived for all constitutive coefficients that can be calculated by numerical solution of certain canonical cell problems. For isotropic and homogeneous media the correction term is found to be cubic in the seepage velocity, hence remains small even for Reynolds numbers which are not very small. This implies that inertia, if it is weak, is of greater importance locally than globally. Existing empirical knowledge is qualitatively consistent with our conclusion since the linear law of Darcy is often accurate for moderate flow rates.

1. Introduction

In flows through a rigid porous medium the celebrated law of Darcy, which is a linear relation between the averaged seepage velocity and the pressure gradient on the macroscale, is well established experimentally for sufficiently small pressure gradients or flow rates. As has been reviewed by many authors (e.g. Rose 1945; Scheidegger 1974; Dullien 1979; Kovacs 1981; Hannoura & Barends 1981), the law is generally regarded as valid even for not so small Reynolds numbers, defined by $R = ul/\nu$ where u is seepage velocity, l typical pore size, and ν kinematic viscosity. Much effort has been focused on the upper limit of Darcy's law and the effect of large R on the permeability. Empirical formulae aiming at a universal expression over a wide range of R less than 1000 have been proposed for isotropic and homogeneous media in the following form due to Forcheimer (1901):

$$-K\frac{\partial\langle p\rangle}{\partial x} = \langle u\rangle + b\langle u\rangle^m \tag{1.1}$$

which modifies Darcy's law through the last nonlinear term. The coefficient K, which is proportional to the hydraulic conductivity of the medium, and b are material constants depending on the pore size, shape and porosity, and the power m is close to 2. The threshold value of R when the nonlinear term becomes appreciable ranges from 0.1 to 75 (Scheidegger 1974, p. 154), but the smaller range of $1 < R < 10$ is generally accepted in the literature. The wide spread is due in part to the uncertainty of l and difficulty in characterizing the pore shape in a natural material. There have also been theoretical attempts to justify (1.1) (see e.g. Bear 1972; Scheidegger 1974; Happel & Brenner 1983; Cvetkovic 1986 for reviews and further references). In these older theories there have been two approaches. In the first, one either models the pores by an array of small tubes, or the solid matrix by an array of fibres, so that the interstitial flow is treated one-dimensionally. In the second approach, one begins by performing Reynolds averaging of Navier–Stokes equations, as in the theory of turbulence. But then a closure hypothesis is added (e.g. Cvetkovic 1986, p. 77) based sometimes only on dimensional reasoning (De Wiest 1969). In this way some coefficients are not determined by the theory itself.

To be sure, even for negligible inertia a strict theory which predicts, without closure assumptions, not only the form of Darcy's law but the permeability is not a simple matter if the complexity of the pore is realistically modelled. The difficulties can be seen in the available approximate analytical theories for simple geometries such as a periodic array of spheres or of circular cylinders, as surveyed by Happel & Brenner (1983). Brenner (1964) has given a general treatment which provides a formal basis for deducing from microscale mechanics the permeability tensor (called by the author the *grand resistance matrix*) in Darcy's law for an aggregate of rigid particles. More recently an alternative and powerful formalism called the *theory of homogenization* (Bensoussan, Lions & Papanicolaou 1978) has been developed which is very convenient for composite media with microscale structures consisting of periodic cells.† Phenomenological equations are derived on the basis of microscopic mechanics by the perturbation method of two scales. In the present problem the two lengthscales are just the typical pore size l and the macroscale L which characterizes the averaged flow, with $l/L = \epsilon \ll 1$. For this model of microstructures, which may be quite adequate for some materials though still highly idealized for some others, there is no need to add closure assumptions and the permeability tensor can be rigorously constructed from the numerical solution of a boundary-value problem defined for a periodic cell on the l-scale. Sample papers are Ene & Sanchez-Palencia (1975) and Keller (1980) for rigid porous media, Auriault & Sanchez-Palencia (1977) for the quasi-static motion, and Levy (1979), Auriault (1980), and Burridge & Keller (1981) for the dynamics of saturated and deformable porous media. The theory of homogenization has also been applied, for the deduction of phenomenological theories from microscale foundations, to many other branches of applied physics involving composite media and microscale heterogeneities (Bensoussan *et al.* 1978; Sanchez-Palencia 1980; Bakhvalov & Panasenko 1989). It has also been extended by us for rigid and deformable porous media with several disparate scales of inhomogeneities (Mei & Auriault 1989). The model of periodic microstructures is also at the base of the theory of dispersion in porous media by Brenner (1980) and Brenner & Adler (1982).

A fully predictive theory for $R = O(1)$ corresponding to (1.1) is difficult in general. As shown by Ene & Sanchez-Palencia (1975) and by Keller (1980), the cell problem involves the full Navier–Stokes equations. Accurate numerical solution for the two-dimensional problem of a corrugated tube with a circular cross section has been given by Payatakes, Tien & Turian (1973). Since the complete solution to the linearized cell problem for packed uniform spheres was derived only recently (Zick & Homsy 1982), it is likely that a fully nonlinear theory for three-dimensional geometries is merely formal at present. In view of this we pursue a less ambitious goal and derive a higher-order theory for rigid porous media when the fluid inertia is small but finite, i.e. $R \ll 1$. By employing the theory of homogenization results are first developed for a general anisotropic medium. All the constitutive coefficients in the macroscopic law

† Periodicity here does not mean that grains or pores must be of uniform size and shape.

will be shown to be defined by certain linear microscale boundary-value problems in a typical cell. These problems involve only Stokes equations and can be solved in the manner of Zick & Homsy. For the important special case of an isotropic and homogeneous medium we shall show that the Darcy law becomes

$$-K\frac{\partial\langle p\rangle}{\partial x_i} = \langle u_i\rangle + b\langle u_i\rangle\langle u_k\rangle\langle u_k\rangle, \tag{1.2}$$

i.e. $n = 3$. The new coefficient b arising from weak inertia will also be proved to be non-negative. The one-dimensional model of parallel corrugated tubes also leads to (1.2), although the case is anisotropic. In general terms quadratic in u_i must be included. That the correction term in (1.2) is cubic in $\langle u_i\rangle$ for small $\langle u_i\rangle$ is consistent with the empirical knowledge that the linear law of Darcy is valid even for $R = O(1)$, and does not contradict (1.1) where the nonlinear term is significant only for large $\langle u_i\rangle$.

2. Formulation

We consider a rigid porous medium with an incompressible Newtonian fluid of constant density. Everywhere in the pores Navier–Stokes equations apply:

$$\frac{\partial u_i}{\partial x_i} = 0, \tag{2.1}$$

$$\rho u_j\frac{\partial u_i}{\partial x_j} = -\frac{\partial p}{\partial x_i} + \mu\nabla^2 u_i. \tag{2.2}$$

On the wetted surface of the solid matrix Γ

$$u_i = 0 \quad \text{on} \quad \Gamma. \tag{2.3}$$

For slow flows we anticipate that the two terms on the right-hand side of (2.2) are equally important. Because of the two contrasting scales the hydrodynamic pressure consists of two parts: the global pressure which has the lengthscale L and the local pressure modification which has the lengthscale l, with $l/L = \epsilon \ll 1$. Let us allow for generality that the two have comparable pressure gradients, then the global pressure must be much greater than the local pressure by the factor $O(1/\epsilon)$. Hence we may regard the former as the driving and the latter the responding pressure. Equating the order of magnitudes of the global pressure gradient to the local viscous stress, we get

$$O(u) \sim l^2 p/\mu L. \tag{2.4}$$

If we use l to normalize formally all space coordinates then

$$\frac{\partial p/\partial x_i}{\mu\nabla^2 u_i} = O(\epsilon^{-1}) \gg 1. \tag{2.5}$$

On the other hand the ratio of inertia to viscous stress (Reynolds number) is of the order

$$O(\delta) = \rho u_j\frac{\partial u_i/\partial x_j}{\mu\nabla^2 u_i} = O\left(\frac{\epsilon\rho l^2 p}{\mu^2}\right). \tag{2.6}$$

Since ϵ can be extremely small, say less than 0.01, for small Reynolds numbers it is of greater interest to consider

$$1 \gg \delta \gg \epsilon \quad \text{or} \quad \frac{\rho l^2 p}{\mu^2} = O\left(\frac{\delta}{\epsilon}\right) \gg 1. \tag{2.7}$$

To avoid cumbersome notation in a two-parameter expansion, we shall specify

$$\epsilon = \delta^2 \tag{2.8}$$

and rewrite (2.2) formally as

$$\delta^3 \rho u_j \frac{\partial u_i}{\partial x_j} = -\frac{\partial p}{\partial x_i} + \delta^2 \mu \nabla^2 u_i. \tag{2.9}$$

Despite its appearance above, the dominant pressure gradient is in fact of $O(\delta^2)$.

The boundary condition on the wetted surface Γ of the pores has already been given by (2.3).

Let us assume in addition that the porous matrix has a periodic microstructure. Each periodic cell Ω is a rectangular box of dimension $O(l)$. We then expect u_i and p to be spatially periodic from cell to cell.

We now introduce the multiple-scale coordinates

$$x_i, \quad X_i = \delta^2 x_i \tag{2.10}$$

and the perturbation expansions

$$\left.\begin{aligned} u_i &= u_i^{(0)} + \delta u_i^{(1)} + \delta^2 u_i^{(2)} + \delta^3 u^{(3)} + \dots, \\ p &= p^{(0)} + \delta p^{(1)} + \delta^2 p^{(2)} + \delta^3 p^{(3)} + \dots, \end{aligned}\right\} \tag{2.11}$$

where $u^{(j)}, p^{(j)}$ are functions of x_i and X_i.

From (2.1) we get, at orders from $O(\epsilon^0)$ to $O(\epsilon^4)$:

$$\frac{\partial u_i^{(0)}}{\partial x_i} = 0, \quad \frac{\partial u_i^{(1)}}{\partial x_i} = 0, \tag{2.12a, b}$$

$$\frac{\partial u_i^{(2)}}{\partial x_i} + \frac{\partial u^{(0)}}{\partial X_i} = 0, \quad \frac{\partial u_i^{(3)}}{\partial x_i} + \frac{\partial u_i^{(1)}}{\partial X_i} = 0, \tag{2.12c, d}$$

$$\frac{\partial u_i^{(4)}}{\partial x_i} + \frac{\partial u_i^{(2)}}{\partial X_i} = 0. \tag{2.12e}$$

Similarly we get from (2.9)

$$0 = -\frac{\partial p^{(0)}}{\partial x_i}, \quad 0 = -\frac{\partial p^{(1)}}{\partial x_i}, \tag{2.13a, b}$$

$$0 = -\frac{\partial p^{(0)}}{\partial X_i} - \frac{\partial p^{(2)}}{\partial x_i} + \mu \nabla^2 u_i^{(0)}, \tag{2.13c}$$

$$\rho u_j^{(0)} \frac{\partial u_i^{(0)}}{\partial x_j} = -\frac{\partial p^{(1)}}{\partial X_i} - \frac{\partial p^{(3)}}{\partial x_i} + \mu \nabla^2 u_i^{(1)}, \tag{2.13d}$$

$$\rho\left(u_j^{(0)} \frac{\partial u_i^{(1)}}{\partial x_j} + u_j^{(1)} \frac{\partial u_i^{(0)}}{\partial x_j}\right) = -\frac{\partial p^{(2)}}{\partial X_i} - \frac{\partial p^{(4)}}{\partial x_i} + \mu \nabla^2 u_i^{(2)} + 2\mu \frac{\partial^2 u_i^{(0)}}{\partial X_k \partial x_k}. \tag{2.13e}$$

On the pore surfaces Γ the velocity vanishes, hence

$$u_i^{(0)} = u_i^{(1)} = u_i^{(2)} = \dots = 0 \quad \text{on} \quad \Gamma. \tag{2.14}$$

In a typical Ω-cell we impose the condition that the flow is periodic, i.e.

$$u_i^{(0)}\, u_i^{(1)} \ldots p^{(0)} p^{(1)} \ldots \text{ are } \Omega\text{-periodic.} \tag{2.15}$$

We shall seek the macroscale equations for the averaged physical quantities up to $O(\delta^2)$.

It may be pointed out that the assumption (2.7) is crucial here. Had it been assumed that the Reynolds number is of order unity, then in (2.9) both δ-factors would be replaced by ϵ; (2.13c) would be augmented by the convective inertia and become fully nonlinear (Ene & Sanchez-Palencia 1975).

3. The first-order problem

From (2.13a, b) we conclude that

$$p^{(0)} = p^{(0)}(X_i), \quad p^{(1)} = p^{(1)}(X_i). \tag{3.1}$$

Because of the linearity of (2.12a) and (2.13c) we can represent $u_i^{(0)}$ and $p^{(2)}$ formally by

$$u_i^{(0)} = -K_{ij} \frac{\partial p^{(0)}}{\partial X_j}, \quad p^{(2)} = -A_j \frac{\partial p^{(0)}}{\partial X_j} + \bar{p}^{(2)}, \tag{3.2}$$

where $\bar{p}^{(2)}(X_i)$ is independent of x_i. It then follows that $K_{ij}(x_i, X_i)$ and $A_j(x_i, X_i)$ must satisfy

$$\frac{\partial K_{ij}}{\partial x_i} = 0, \tag{3.3}$$

$$-\frac{\partial A_j}{\partial x_i} + \mu \nabla^2 K_{ij} = -\delta_{ij}, \tag{3.4}$$

where

$$K_{ij} = 0 \quad \text{on} \quad \Gamma, \tag{3.5}$$

$$K_{ij}, A_j \text{ are } \Omega\text{-periodic.} \tag{3.6}$$

This defines a Stokes flow boundary-value problem in an Ω-cell. The existence and uniqueness of $u_i^{(0)}$ and $\nabla p^{(2)}$ of this cell problem has been established by Ene & Sanchez-Palencia (1975) and the solution can be obtained numerically for any prescribed microstructure. In particular, for a porous matrix composed of uniform spheres, the numerical problem has been solved by Zick & Homsy (1982, where earlier references may be found).

Defining the average over an Ω-cell by

$$\langle f \rangle = \frac{1}{|\Omega|} \int_{\Omega_f} f \, \mathrm{d}\Omega, \tag{3.7}$$

where Ω_f is the fluid volume inside the Ω-cell, we get

$$\langle u_i^0 \rangle = -\langle K_{ij} \rangle \frac{\partial p^{(0)}}{\partial X_j}, \tag{3.8}$$

$$\langle p^{(2)} \rangle = -\langle A_j \rangle \frac{\partial p^{(0)}}{\partial X_j} + n \bar{p}^{(2)}, \tag{3.9}$$

where n is the porosity

$$n = |\Omega_f| / |\Omega|. \tag{3.10}$$

Equation (3.8) is just Darcy's law with the permeability tensor $\langle K_{ij}\rangle$ determinable theoretically from the cell problem. The Ω-average of (2.12c) gives

$$\frac{\partial \langle u_i^{(0)}\rangle}{\partial X_i}=0 \tag{3.11}$$

after invoking Gauss' theorem and the boundary conditions on Γ and the boundaries of Ω. This implies, in turn,

$$\frac{\partial}{\partial X_i}\langle K_{ij}\rangle \frac{\partial p^{(0)}}{\partial X_j}=0. \tag{3.12}$$

Equations (3.8) and (3.10) or (3.11) are of course well known.

From the cell boundary-value problem Ene & Sanchez-Palencia have further shown that $\langle K_{ij}\rangle$ is symmetric,

$$\langle K_{ij}\rangle=\langle K_{ji}\rangle \tag{3.13}$$

and that $\langle K_{ij}\rangle$ is positive definite.

For the special case where the medium is isotropic and homogeneous on the L-scale, we have

$$\langle K_{ij}\rangle = K\delta_{ij}, \quad \langle A_j\rangle = 0, \tag{3.14}$$

where K is a constant. It follows from (3.12) that

$$\frac{\partial^2 p^{(0)}}{\partial X_k\,\partial X_k}=0. \tag{3.15}$$

With proper boundary conditions for $p^{(0)}$ on the macroscale, $p^{(0)}$ can then be found.

4. The second-order problem

We now consider the cell problem for $u_i^{(1)}$ and $p^{(3)}$ defined by (2.12b) and (2.13d) and the boundary conditions. The forcing function in (2.13d) is

$$\rho u_g^{(0)}\frac{\partial u_i^{(0)}}{\partial x_g}+\frac{\partial p^{(1)}}{\partial X_i}=\rho K_{gj}\frac{\partial K_{ik}}{\partial x_g}\left(\frac{\partial p^{(0)}}{\partial X_k}\frac{\partial p^{(0)}}{\partial X_j}\right)+\frac{\partial p^{(1)}}{\partial X_i}. \tag{4.1}$$

We therefore assume that

$$\begin{pmatrix} u_i^{(1)} \\ p^{(3)}\end{pmatrix} = -\begin{pmatrix} L_{ijk} \\ B_{jk}\end{pmatrix}\frac{\partial p^{(0)}}{\partial X_j}\frac{\partial p^{(0)}}{\partial X_k}-\begin{pmatrix} K_{ij} \\ A_j\end{pmatrix}\frac{\partial p^{(1)}}{\partial X_j}+\begin{pmatrix} 0 \\ \bar{p}^{(3)}(X_i)\end{pmatrix}. \tag{4.2}$$

It follows by substituting (4.2) into (2.12b) and (2.13d) that the new coefficient tensors must be governed by

$$\frac{\partial L_{ijk}}{\partial x_i}=0, \tag{4.3}$$

$$-\frac{\partial B_{jk}}{\partial x_i}+\mu\nabla^2 L_{ijk}=-\rho K_{gj}\frac{\partial K_{ik}}{\partial x_g}, \tag{4.4}$$

with the boundary conditions that

$$L_{ijk}=0 \quad \text{on} \quad \Gamma, \tag{4.5}$$

$$L_{ijk},\ B_{jk} \quad \text{are} \quad \Omega\text{-periodic}. \tag{4.6}$$

The cell problem defined by (4.3)–(4.6) is of the same type as that for K_{ij} and A_j and can be solved by the same numerical procedure, and the resulting Ω-averages gives

$$\begin{pmatrix} \langle u_i^{(1)}\rangle \\ \langle p^{(3)}\rangle\end{pmatrix} = -\begin{pmatrix} \langle L_{ijk}\rangle \\ \langle B_{jk}\rangle\end{pmatrix}\frac{\partial p^{(0)}}{\partial X_j}\frac{\partial p^{(0)}}{\partial X_k}-\begin{pmatrix} \langle K_{jk}\rangle \\ \langle A_j\rangle\end{pmatrix}\frac{\partial p^{(1)}}{\partial X_j}+\begin{pmatrix} 0 \\ n\bar{p}^{(3)}\end{pmatrix}. \tag{4.7}$$

The Ω-average of (2.12*d*) gives

$$\frac{\partial \langle u_i^{(1)} \rangle}{\partial X_i} = 0 \tag{4.8}$$

after using Gauss' theorem and the secondary conditions on $u_i^{(2)}$; this in turn implies a governing equation for $p^{(1)}$:

$$\frac{\partial}{\partial X_i}\left\{\langle K_{ij}\rangle \frac{\partial p^{(1)}}{\partial X_j}\right\} = -\frac{\partial}{\partial X_i}\left\{\langle L_{ijk}\rangle \frac{\partial p^{(0)}}{\partial X_j}\frac{\partial p^{(0)}}{\partial X_k}\right\}. \tag{4.9}$$

When the medium is isotropic and homogeneous with respect to X_i, $\langle L_{ijk}\rangle$ must be proportional to the permutation tensor ϵ_{ijk} which has the properties that

$$\epsilon_{123} = \epsilon_{231} = \epsilon_{312} = 1, \quad \epsilon_{132} = \epsilon_{321} = \epsilon_{213} = -1 \tag{4.10}$$

and all other components vanish. It follows that

$$\langle L_{ijk}\rangle \frac{\partial p^{(0)}}{\partial X_j}\frac{\partial p^{(0)}}{\partial X_k} = L(\epsilon_{123}+\epsilon_{132})\frac{\partial p^{(0)}}{\partial X_2}\frac{\partial p^{(0)}}{\partial X_3}$$
$$+L(\epsilon_{231}+\epsilon_{213})\frac{\partial p^{(0)}}{\partial X_1}\frac{\partial p^{(0)}}{\partial X_3}+L(\epsilon_{312}+\epsilon_{321})\frac{\partial p^{(0)}}{\partial X_1}\frac{\partial p^{(0)}}{\partial X_2} = 0. \tag{4.11}$$

On the other hand
$$\langle B_{jk}\rangle = B\delta_{jk}, \tag{4.12}$$

where B is a constant. Further, $p^{(1)}(X_i) = n^{-1}\langle p^{(1)}\rangle$ satisfies (3.15). Without loss of generality, one may then absorb $p^{(1)}$ in $p^{(0)}$, or take

$$p^{(1)} = 0, \tag{4.13}$$

which implies that $\langle u_i^{(1)}\rangle$ may also be taken to be zero.

A more general identity can be derived. Scalar-multiplying (3.4) by L_{iqk}, we get

$$-L_{jqk} = -\frac{\partial}{\partial x_i}(L_{iqk}A_j) + A_j\frac{\partial L_{iqk}}{\partial x_i} + \frac{\partial}{\partial x_g}\left(L_{iqk}\frac{\partial K_{ig}}{\partial x_g}\right) - \frac{\partial L_{iqk}}{\partial x_g}\frac{\partial K_{ij}}{\partial x_g}. \tag{4.14}$$

Clearly, by integrating the preceding equation over the Ω-cell and invoking (3.3), (3.4) and the boundary conditions, we get

$$-\int_\Omega \mathrm{d}\Omega\, L_{jqk} = \int_\Omega \mathrm{d}\Omega \frac{\partial L_{iqk}}{\partial x_g}\frac{\partial K_{ij}}{\partial x_g}. \tag{4.15}$$

Similarly we find by scalar-multiplying (4.4) by K_{ij} and integrating over the Ω-cell,

$$-\rho\int_\Omega \mathrm{d}\Omega\, K_{ij}K_{gq}\frac{\partial K_{ik}}{\partial x_g} = -\int_\Omega \mathrm{d}\Omega\frac{\partial K_{ij}}{\partial x_g}\frac{\partial L_{iqk}}{\partial x_g}. \tag{4.16}$$

It follows from the difference of (4.14) and (4.16) that

$$\int_\Omega \mathrm{d}\Omega\, L_{jqk} = \rho\int_\Omega \mathrm{d}\Omega\, K_{ij}K_{gq}\frac{\partial K_{ik}}{\partial x_g}, \tag{4.17}$$

which can be written alternatively as

$$\int_\Omega \mathrm{d}\Omega\, L_{kqj} = \rho\int_\Omega \mathrm{d}\Omega\, K_{ik}K_{gq}\frac{\partial K_{ij}}{\partial x_g} \tag{4.18}$$

after interchanging j and k. Adding both sides of the above two equations and noting that

$$\rho\int_{\Omega} \mathrm{d}\Omega\left(K_{ij}K_{gq}\frac{\partial K_{ik}}{\partial x_g}+K_{ik}K_{gq}\frac{\partial K_{ij}}{\partial x_g}\right) = \rho\int_{\Omega}\mathrm{d}\Omega\left[\frac{\partial}{\partial x_g}(K_{ij}K_{gq}K_{ik})-K_{ij}K_{ik}\frac{\partial K_{gq}}{\partial x_g}\right] = 0, \tag{4.19}$$

we obtain the following identity:

$$\langle L_{kqj}\rangle + \langle L_{jqk}\rangle = 0. \tag{4.20}$$

In the special case where the global pressure gradient is in one direction only, say X_1,

$$\frac{\partial p^{(0)}}{\partial X_1} \neq 0, \quad \frac{\partial p^{(0)}}{\partial X_2} = \frac{\partial p^{(0)}}{\partial X_3} = 0, \tag{4.21}$$

then

$$\langle L_{111}\rangle = 0, \quad \text{implying} \quad \langle u_1^{(1)}\rangle = 0. \tag{4.22}$$

However, $\langle L_{211}\rangle$ and $\langle L_{311}\rangle$ are in general non-zero, hence $\langle u_2^{(1)}\rangle$ and $\langle u_3^{(1)}\rangle$ are also non-zero, and the quadratic correction to Darcy's law is anisotropic.

5. The third-order problem

For $u_i^{(2)}$ and $p^{(4)}$ the governing equations in a Ω-cell are (2.12c) and (2.13e). When (3.2) is used, (2.14c) gives

$$\frac{\partial u_i^{(2)}}{\partial x_i} = \frac{\partial}{\partial X_j}\left(K_{jk}\frac{\partial p^{(0)}}{\partial X_k}\right) = \frac{\partial K_{jk}}{\partial X_j}\frac{\partial p^{(0)}}{\partial X_k}+K_{jk}\frac{\partial^2 p^{(0)}}{\partial X_j\partial X_k}. \tag{5.1}$$

Similarly by using (3.2) and (4.2), (2.13e) becomes

$$\begin{aligned}-\frac{\partial p^{(4)}}{\partial x_i}+\mu\nabla^2 u_i^{(2)} = {}&+\frac{\partial \bar{p}^{(2)}}{\partial X_i}+\rho\left(K_{lg}\frac{\partial L_{ijk}}{\partial x_l}+L_{ljk}\frac{\partial K_{ij}}{\partial x_l}\right)\frac{\partial p^{(0)}}{\partial X_j}\frac{\partial p^{(0)}}{\partial X_k}\frac{\partial p^{(0)}}{\partial X_g}\\ &+\rho K_{lk}\frac{\partial K_{ij}}{\partial x_l}\left(\frac{\partial p^{(0)}}{\partial X_k}\frac{\partial p^{(1)}}{\partial X_j}+\frac{\partial p^{(1)}}{\partial X_k}\frac{\partial p^{(0)}}{\partial X_j}\right)-\delta_{ij}\frac{\partial A_k}{\partial X_j}\frac{\partial p^{(0)}}{\partial X_k}\\ &-A_k\delta_{ij}\frac{\partial^2 p^{(0)}}{\partial X_k\partial X_j}+2\mu\left(\frac{\partial}{\partial X_j}\frac{\partial K_{ik}}{\partial x_j}\right)\frac{\partial p^{(0)}}{\partial X_k}+2\mu\frac{\partial K_{ik}}{\partial x_j}\frac{\partial^2 p^{(0)}}{\partial X_j\partial X_k}.\end{aligned} \tag{5.2}$$

For brevity we introduce the following notation for the forcing term:

$$\left.\begin{aligned} F'_{ijkg} &= \rho\left(K_{lg}\frac{\partial L_{ijk}}{\partial x_l}+L_{ljk}\frac{\partial K_{ig}}{\partial x_l}\right),\\ F''_{ijk} &= -A_k\delta_{ij}+2\mu\frac{\partial K_{ik}}{\partial x_j},\\ F'''_{ijk} &= \rho K_{lk}\frac{\partial K_{ij}}{\partial x_l},\\ F^{iv}_{ik} &= -\delta_{ij}\frac{\partial A_k}{\partial X_j}+2\mu\frac{\partial}{\partial X_j}\left(\frac{\partial K_{ik}}{\partial x_j}\right),\end{aligned}\right\} \tag{5.3}$$

and then express the solution as

$$\begin{pmatrix} u_i^{(2)} \\ p^{(4)} \end{pmatrix} = -\begin{pmatrix} K_{ij} \\ A_j \end{pmatrix} \frac{\partial \bar{p}^{(2)}}{\partial X_j} - \begin{pmatrix} M'_{ijkg} \\ C'_{jkg} \end{pmatrix} \frac{\partial p^{(0)}}{\partial X_j} \frac{\partial p^{(0)}}{\partial X_k} \frac{\partial p^{(0)}}{\partial X_g} - \begin{pmatrix} M''_{ijk} \\ C''_{jk} \end{pmatrix} \frac{\partial^2 p^{(0)}}{\partial X_j \partial X_k}$$
$$- \begin{pmatrix} M'''_{ijk} \\ C'''_{jk} \end{pmatrix} \left(\frac{\partial p^{(0)}}{\partial X_k} \frac{\partial p^{(1)}}{\partial X_j} + \frac{\partial p^{(1)}}{\partial X_k} \frac{\partial p^{(0)}}{\partial X_j} \right) - \begin{pmatrix} M^{\mathrm{iv}}_{ik} \\ C^{\mathrm{iv}}_{k} \end{pmatrix} \frac{\partial p^{(0)}}{\partial X_k} + \begin{pmatrix} 0 \\ \bar{p}^{(4)} \end{pmatrix}, \quad (5.4)$$

where $\bar{p}^{(4)} = \bar{p}^{(4)}(X_i)$. Then from (5.1) and (5.2) the governing equations for the new coefficient tensors are

$$\frac{\partial M'_{ijkg}}{\partial x_i} = 0, \quad -\frac{\partial C'_{jkg}}{\partial x_i} + \mu \nabla^2 M'_{ijkg} = -F'_{ijkg}, \quad (5.5a, b)$$

$$\frac{\partial M''_{ijk}}{\partial x_i} = -K_{jk}, \quad -\frac{\partial C''_{jk}}{\partial x_i} + \mu \nabla^2 M''_{ijk} = -F''_{ijk}, \quad (5.5c, d)$$

$$\frac{\partial M'''_{ijk}}{\partial x_i} = 0, \quad -\frac{\partial C'''_{jk}}{\partial x_i} + \mu \nabla^2 M'''_{ijk} = -F'''_{ijk}, \quad (5.5e, f)$$

$$\frac{\partial M^{\mathrm{iv}}_{ik}}{\partial x_i} = -\frac{\partial K_{jk}}{\partial X_j}, \quad -\frac{\partial C^{\mathrm{iv}}_{k}}{\partial x_i} + \mu \nabla^2 M^{\mathrm{iv}}_{ik} = -F^{\mathrm{iv}}_{ik}. \quad (5.5g, h)$$

For each of the four sets of unknowns the no-slip and periodicity conditions must be added. Thus we have four similar cell problems which can again be solved numerically. The Ω-average of (5.4) gives $\langle u^{(2)} \rangle$ and $\langle p^{(4)} \rangle$ in terms of the macroscale gradients of $p^{(0)}$, $p^{(1)}$ and $\bar{p}^{(2)}$ and the averaged coefficients in (5.4). To find $\bar{p}^{(2)}$ we take the Ω-average of (2.13*e*) to get

$$\frac{\partial \langle u_i^{(2)} \rangle}{\partial X_i} = 0, \quad (5.6)$$

which in turn gives the governing equation for $\bar{p}^{(2)}$:

$$\frac{\partial}{\partial X_i} \left\{ \langle K_{ij} \rangle \frac{\partial \bar{p}^{(2)}}{\partial X_j} \right\} = \frac{\partial}{\partial X_i} \left[-\langle M'_{ijkg} \rangle \frac{\partial p^{(0)}}{\partial X_j} \frac{\partial p^{(0)}}{\partial X_k} \frac{\partial p^{(0)}}{\partial X_g} - \langle M''_{ijk} \rangle \frac{\partial^2 p^{(0)}}{\partial X_j \partial X_k} \right.$$
$$\left. - \langle M'''_{ijk} \rangle \left(\frac{\partial p^{(0)}}{\partial X_k} \frac{\partial p^{(1)}}{\partial X_j} + \frac{\partial p^{(1)}}{\partial X_k} \frac{\partial p^{(0)}}{\partial X_j} \right) - \langle M^{\mathrm{iv}}_{ik} \rangle \frac{\partial p^{(0)}}{\partial X_k} \right]. \quad (5.7)$$

For a general anisotropic and inhomogeneous medium, the field equations for the macroscale quantities $\langle u_i \rangle = \langle u_i^{(0)} \rangle + \delta \langle u_i^{(1)} \rangle + \delta^2 \langle u_i^{(2)} \rangle$ and $\langle p \rangle = p^{(0)} + \delta p^{(1)} + \delta^2 \langle p^{(2)} \rangle$ are now known. With proper boundary conditions according to the physical situations, the boundary-value problems are now complete up to the order δ^2.

We now restrict our attention to media that are isotropic and homogeneous with respect to X_i. First

$$M^{\mathrm{iv}}_{ik} = \frac{\partial C^{\mathrm{iv}}_{k}}{\partial x_i} \equiv 0 \quad (5.8)$$

because all the forcing terms in (5.5, *g*, *h*) vanish by assumption of homogeneity in X_i. There is no contribution to the Ω-average of (5.4) by M'''_{ijk} and C'''_{jk} because $\bar{p}^{(1)} = 0$ (see (4.13)). The Ω-averages of the third-rank isotropic tensors $\langle M'''_{ijk} \rangle$ and $\langle C'''_{ijk} \rangle$ are proportional to the permutation tensors; and the associated terms in (5.4) cancel in pairs, as in (4.11). The most general isotropic tensor of rank four is

$$\langle M'_{ijkg} \rangle = \lambda \delta_{ij} \delta_{kg} + \mu (\delta_{ik} \delta_{jg} + \delta_{ig} \delta_{jk}) + \nu (\delta_{ik} \delta_{jg} - \delta_{ig} \delta_{jk}), \quad (5.9)$$

where λ, μ and ν are scalars. Using these facts we obtain, simply,

$$\langle u_i^{(2)}\rangle = -K\frac{\partial \bar{p}^{(2)}}{\partial X_i} - \beta\frac{\partial p^{(0)}}{\partial X_i}\frac{\partial p^{(0)}}{\partial X_k}\frac{\partial p^{(0)}}{\partial X_k}, \tag{5.10}$$

$$\langle p^{(4)}\rangle = -C''\frac{\partial p^{(0)}}{\partial X_k}\frac{\partial p^{(0)}}{\partial X_k} + \bar{p}^{(4)}, \tag{5.11}$$

where β is a constant coefficient defined by

$$\beta = \lambda + 2\mu = \langle M'_{iiii}\rangle \quad \text{(no summation over } i\text{)}, \tag{5.12}$$

and C'' is defined by
$$C''_{ik} = C''\delta_{ik}. \tag{5.13}$$

Combining (5.10) and (5.6) we obtain the governing equation for $\bar{p}^{(2)}$:

$$K\frac{\partial^2 \bar{p}^{(2)}}{\partial X_i \partial X_i} = -\beta\frac{\partial}{\partial X_i}\left(\frac{\partial p^{(0)}}{\partial X_i}\frac{\partial p^{(0)}}{\partial X_k}\frac{\partial p^{(0)}}{\partial x_k}\right). \tag{5.14}$$

We recall that the permeability coefficient K is found from the cell problem (3.3)–(3.6), and β from (5.5*a*, *b*) and the boundary conditions of no slip and periodicity.

6. Summary for an isotropic homogeneous medium

By combining (3.8), (4.7) and (5.10) we obtain

$$\begin{aligned}\langle u_i\rangle &= \langle u_i^{(0)}\rangle + \delta\langle u_i^{(1)}\rangle + \delta^2\langle u_i^{(2)}\rangle \\ &= -K\left(\frac{\partial p^{(0)}}{\partial X_i} + \delta^2\frac{\partial \bar{p}^{(2)}}{\partial X_i}\right) - \delta^2\beta\frac{\partial p^{(0)}}{\partial X_i}\frac{\partial p^{(0)}}{\partial X_k}\frac{\partial p^{(0)}}{\partial X_k} + O(\delta^3).\end{aligned} \tag{6.1}$$

Similarly by combining (3.9) with $\langle A_j\rangle = 0$ we have

$$\langle p\rangle = p^{(0)} + \delta^2\bar{p}^{(2)} + O(\delta^3). \tag{6.2}$$

Equation (6.1) can be rewritten as

$$\langle u_i\rangle = -K\frac{\partial\langle p\rangle}{\partial X_i} - \delta^2\beta\frac{\partial\langle p\rangle}{\partial X_i}\frac{\partial\langle p\rangle}{\partial X_k}\frac{\partial\langle p\rangle}{\partial X_k} + O(\delta^3). \tag{6.3}$$

To the same order of accuracy one can alternatively write

$$\langle u_i\rangle\left(1 - \frac{\delta^2\beta}{K^3}\langle u_k\rangle\langle u_k\rangle\right) = -K\frac{\partial\langle p\rangle}{\partial X_i} + O(\delta^3). \tag{6.4}$$

Either (6.3) or (6.4) is the extension to Darcy's law for low Reynolds numbers, and should be combined with

$$\frac{\partial\langle u_i\rangle}{\partial X_i} = 0 \tag{6.5}$$

to complete the macroscale equations.

Thus for a low-Reynolds-number flow in an isotropic homogeneous medium, the leading-order correction due to convective inertia is a term cubic in the mean velocity. The coefficients K and β are completely determined by their respective cell problems. More important, (6.4) implies that the linear law of Darcy is very accurate even for Reynolds numbers not much smaller than unity. In fact, available experimental data indicate that the domain of accuracy extends further to $0 < R < 1$. For example, based on a comprehensive analysis of four different studies for homodisperse spheres, Kovacs (1981) plotted the effective coefficient K_{eff}, defined by

$$\frac{1}{K_{\mathrm{eff}}} = -\frac{\partial\langle p\rangle}{\partial X}\frac{1}{\langle u\rangle}, \tag{6.6}$$

versus the Reynolds number, for $0.06 < R < 1000$. Among the data he collected, the majority is for high Reynolds numbers, though some (due to Zunker 1930) are for low Reynolds numbers. (For $R < 0.1$, only one data point is available for sphere diameter $= 0.93$ mm ($n = 0.374$), at $R = 0.065$. In the range $0.1 < R < 0.5$, for each of two diameters equal to 0.93 mm and 0.79 mm ($n = 0.369$) there are only four data points, at $R = 0.13, 0.19, 0.33, 0.43$.) It is, however, unmistakably clear that for these small Reynolds numbers, K_{eff} is virtually independent of R. Thus it is likely that the coefficient β in (6.4) is numerically small for homodisperse spheres. There are also many other experiments done with natural sand and rockfills, but the data scatter is often rather large (see e.g. McCorquodale, Hannoura & Nasser 1978) so that one can only surmise the constancy of the statistical average of K_{eff} for small R. It is not possible to use these data to draw any conclusion on the inertia correction.

7. Proof that $\beta \leqslant 0$

On intuitive grounds one expects that, to maintain the same seepage velocity, a higher pressure gradient is needed if fluid inertia becomes increasingly important. Hence β should be non-positive. This is established theoretically by proving that $\langle M'_{iiii}\rangle \leqslant 0$, as follows.

Changing the index j to p in (3.4), taking the scalar product of the resulting equation with M'_{ijkg}, and integrating over the fluid in the Ω-cell, we obtain after partial integration,

$$-\mu\int_{\Omega_f}\frac{\partial K_{ip}}{\partial x_q}\frac{\partial M'_{ijkg}}{\partial x_q}\,\mathrm{d}\Omega = -\int_{\Omega_f}\delta_{ip}M'_{ijkg}\,\mathrm{d}\Omega = -\int_{\Omega_f}M'_{pjkg}\,\mathrm{d}\Omega. \tag{7.1}$$

Now multiplying (5.5*b*) scalarly by K_{ip} and integrating over Ω_f also we find

$$-\mu\int_{\Omega_f}\frac{\partial M'_{ijkg}}{\partial x_q}\frac{\partial K_{ip}}{\partial x_q}\,\mathrm{d}\Omega = -\int_{\Omega_f}A_{ijkg}K_{ip}\,\mathrm{d}\Omega. \tag{7.2}$$

It follows that

$$\langle M'_{pjkg}\rangle = \langle F'_{ijkg}K_{ip}\rangle = \rho\left\langle\left(K_{lg}\frac{\partial L_{ijk}}{\partial x_l}+L_{ljk}\frac{\partial K_{ig}}{\partial x_l}\right)K_{ip}\right\rangle \tag{7.3}$$

after using (5.3*a*). We need to prove that

$$\langle M'_{pppp}\rangle = \rho\left\langle\left(K_{lp}\frac{\partial L_{ipp}}{\partial x_l}+l_{lpp}\frac{\partial K_{ip}}{\partial x_l}\right)K_{ip}\right\rangle. \tag{7.4}$$

After changing the indices j to g, g to l, and k to p in (4.4), we next multiply the resulting equation by L_{ijk} and integrate over the Ω-cell, to find

$$\mu\int_{\Omega_f}\frac{\partial L_{igp}}{\partial x_q}\frac{\partial L_{ijk}}{\partial x_q}\,\mathrm{d}\Omega = \rho\int_{\Omega_f}K_{lg}\frac{\partial K_{ip}}{\partial x_l}L_{ijk}\,\mathrm{d}\Omega. \tag{7.5}$$

The right-hand side can be manipulated to give

$$\rho \int_{\Omega_f} \left(\frac{\partial}{\partial x_l} K_{lg} K_{ip}\right) L_{ijk}\, \mathrm{d}\Omega = -\rho \int_{\Omega_f} K_{lg} K_{ip} \frac{\partial L_{ijk}}{\partial x_l} \mathrm{d}\Omega \tag{7.6}$$

after partial integration and the use of (3.3).

If we take $j = k = g = p$ in (7.5) and (7.6) without summing over p, the result is

$$\mu \int_{\Omega_f} \frac{\partial L_{ipp}}{\partial x_q} \frac{\partial L_{ipp}}{\partial x_q} \mathrm{d}\Omega = -\rho \int_{\Omega_f} K_{lp} K_{ip} \frac{\partial L_{ipp}}{\partial x_l} \mathrm{d}\Omega, \tag{7.7}$$

which is non-negative. Clearly the first term on the right of (7.4) is non-positive, i.e.

$$\rho \left\langle K_{lp} K_{ip} \frac{\partial L_{ipp}}{\partial x_l} \right\rangle \leqslant 0. \tag{7.8}$$

For the second term on the right of (7.4) we get

$$\rho \left\langle L_{lpp} \frac{\partial K_{ip}}{\partial x_l} K_{ip} \right\rangle = \tfrac{1}{2}\rho \left\langle L_{lpp} \frac{\partial K_{ip}^2}{\partial x_l} \right\rangle = \tfrac{1}{2}\rho \left\langle \frac{\partial}{\partial x_l} (L_{lpp} K_{ip}^2) \right\rangle = 0 \tag{7.9}$$

after using (4.3), Gauss' theorem and the boundary conditions. Finally, (7.8) and (7.9) together imply that

$$\beta = \langle M'_{pppp} \rangle \leqslant 0 \tag{7.10}$$

as expected; therefore $b \geqslant 0$ in (1.2). This does not exclude the possibility that β is close to zero for some geometries, so that convective inertia that must be locally important still does not alter Darcy's law on the macroscale.

8. A one-dimensional model

The most elementary model of a porous medium that can be treated analytically is one with parallel and straight tubes. This model permits flow in one direction only and the fluid mechanics problem reduces to that for a single tube. Scheidegger (1974) also considered parallel tubes consisting of serially connected tube segments of different cross-sections. However, he only accounted for the Poiseille flow in the straight portion of each segment but not the junctions, hence not the convective inertia. In the literature there are also three-dimensional models consisting of a network of tubular segments; the details of the junctions are ignored and only the flow resistance along the straight portions is accounted for (see e.g. Bear 1972) for a survey of earlier literature and Adler & Brenner 1984*a*, *b* who treated non-Newtonian fluids). This omission can be justified at the leading order if the tube radius is much smaller than the typical distance between junctions. The problem then involves three vastly different lengthscales: $a \ll l \ll L$. A leading-order theory for such a porous medium with $a/l \sim l/L \sim O(\delta)$ and $Re = O(\delta)$ has been given by Mei & Auriault (1989) who point out that inertia is important only at $O(\delta^3)$ (see comment after (2.14*c*) therein).

Remaining in the context of two scales, we discuss here a quasi-one-dimensional medium with pores in the form of corrugated tubes parallel to the x-axis. The radius $a(x)$ of a typical tube varies periodically and significantly in x at the wavelength of l, which is of the order a, as sketched in figure 1. For simplicity a is assumed to be

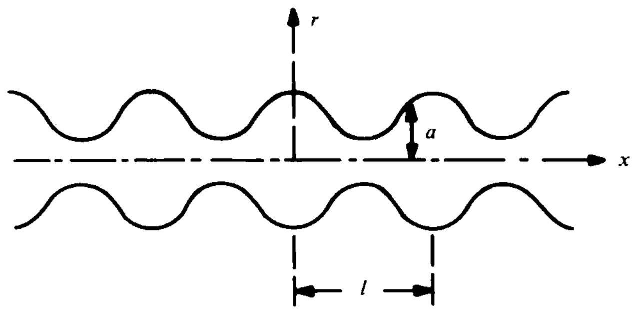

FIGURE 1. A corrugated tube.

independent of the macroscale. The macroscale problem is one-dimensional, homogeneous but anisotropic, and the microscale problems are only two-dimensional in x and r, where r is the radial coordinate measured from the axis of a typical tube.

We denote the longitudinal and radial components of velocity $\boldsymbol{u}$ by u and w respectively. It follows from (3.1) that

$$p^{(0)} = p^{(0)}(X), \quad p^{(1)} = p^{(1)}(X). \tag{8.1}$$

The leading-order velocity field can be written

$$\boldsymbol{u}^{(0)} = -\boldsymbol{K}\frac{\partial p^{(0)}}{\partial X}, \quad p^{(2)} = \bar{p}^{(2)} - A\frac{\partial p^{(2)}}{\partial X}, \tag{8.2a, b}$$

where
$$u^{(0)} = (u^{(0)}, w^{(0)}) \quad \text{and} \quad \boldsymbol{K} = (K_x, K_r). \tag{8.3}$$

The Ω-cell is the interior of a typical tube within one wavelength l. The unknown coefficients K_x, K_r and A are governed by

$$\nabla \cdot \boldsymbol{K} = 0, \tag{8.4}$$

$$-(1,0) = -\nabla A + \mu \nabla^2 \boldsymbol{K} \tag{8.5}$$

and the boundary conditions

$$\boldsymbol{K} = 0 \quad \text{on} \quad r = a(x); \quad \boldsymbol{K}, A: \quad \Omega\text{-periodic}, \tag{8.6}$$

where
$$\nabla \equiv \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial r}\right) \quad \nabla \cdot \equiv \left(\frac{\partial}{\partial x}, \frac{1}{r}\frac{\partial}{\partial r}r\right) \quad \nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r}. \tag{8.7}$$

This axisymmetric Stokes flow problem can be solved numerically by finite elements as in Auriault, Borne & Champon (1985) who treated a two-dimensional problem of rectangular blocks in a plane. From the solution we then take the volume average over a unit cube of length l defined by

$$\langle f \rangle = \frac{n}{\Omega_f}\int_l \mathrm{d}x \int_0^a 2\pi f r\,\mathrm{d}r \tag{8.8}$$

with l also representing the interval $(x, x+l)$ where

$$\Omega_f = \pi \int_l a^2\,\mathrm{d}x \tag{8.9}$$

is the pore volume per period and

$$n = \frac{\pi}{l^3}N\int_l a^2(x)\,\mathrm{d}x \tag{8.10}$$

is the porosity, with N being the number of tubes in a square of area l^2. The result of averaging (8.2a) is

$$\langle \boldsymbol{u}^{(0)} \rangle = -\langle \boldsymbol{K} \rangle \frac{\partial p^{(0)}}{\partial X}. \tag{8.11}$$

From (3.15) we get simply

$$\frac{\partial^2 p^{(0)}}{\partial X^2} = 0; \tag{8.12}$$

thus the macroscale pressure gradient is constant.

On physical grounds we anticipate that

$$\langle K_r \rangle = 0 \quad \text{so that} \quad \langle w^{(0)} \rangle = 0. \tag{8.13}$$

This can be confirmed by first finding K_r from (8.8)

$$rK_r = \int_0^r \frac{\partial}{\partial x}(rK_x)\,\mathrm{d}r \tag{8.14}$$

and integrating over the cell volume

$$\begin{aligned}\int_l \mathrm{d}x \int_0^a \mathrm{d}r\, rK_r &= \int_l \mathrm{d}x \int_0^a \mathrm{d}r\, r\frac{\partial}{\partial x}\int_0^r \mathrm{d}r'\, r'K_r \\ &= \int_l \mathrm{d}x \frac{\partial}{\partial x}\int_0^a \mathrm{d}r\, r\int_0^r \mathrm{d}r'\, r'K_x - \int_l \mathrm{d}x \frac{\partial a}{\partial x}\int_0^a \mathrm{d}r\, rK_x \\ &= \left[a\int_0^a \mathrm{d}r\, rK_x\right]_x^{x+l} - \int_l \mathrm{d}x\, a^2\frac{\partial a}{\partial x}K_x(x, a(x)) = 0,\end{aligned} \tag{8.15}$$

where (8.6) has been invoked. This proves that $\langle K_r \rangle = 0$, hence the macroscale flow is trivially anisotropic to the leading order.

The next-order solution can be written as

$$\boldsymbol{u}(1) = -\boldsymbol{K}\frac{\partial p^{(1)}}{\partial X} - \boldsymbol{L}\left(\frac{\partial p^{(0)}}{\partial X}\right)^2, \tag{8.16}$$

and

$$p^{(3)} = \bar{p}^{(3)}(X) - A\frac{\partial p^{(1)}}{\partial X} - B\left(\frac{\partial p^{(0)}}{\partial X}\right)^2, \tag{8.17}$$

It is easy to see that $p^{(1)}$ satisfies (8.12) also and may be taken to be zero, while $\boldsymbol{L}$ and B satisfy

$$\boldsymbol{\nabla}\cdot\boldsymbol{L} = 0, \tag{8.18}$$

$$-\rho(\boldsymbol{K}\cdot\boldsymbol{\nabla}\boldsymbol{K}) = -\boldsymbol{\nabla}B + \mu\nabla^2\boldsymbol{L} \tag{8.19}$$

and the boundary conditions

$$\boldsymbol{L} = 0, \quad r = a(x); \quad \boldsymbol{L}, B\colon \quad \Omega\text{-periodic}. \tag{8.20}$$

Although it is possible to solve this inhomogeneous Stokes flow problem for the local flow, the Ω-average of the velocity field is however zero. This is because $\langle L_r \rangle = 0$, which can be shown in the same way as for the case $\langle K_r \rangle$, and $\langle L_x \rangle = 0$ because of (4.22).

At the third order we first use (8.11) to get

$$\frac{\partial^2 u^{(2)}}{\partial x\,\partial X} = 0. \tag{8.21}$$

Expressing the velocity and pressure in the form

$$\boldsymbol{u}^{(2)} = -\boldsymbol{M}\left(\frac{\partial p^{(0)}}{\partial X}\right)^3, \quad p^{(4)} = -C\left(\frac{\partial p^{(0)}}{\partial X}\right)^3 \tag{8.22}$$

with

$$\boldsymbol{M} = (M_x, M_r) \tag{8.23}$$

we find that the averaged $\bar{p}^{(2)}$ satisfies (8.15) and can be set to zero. The cell problem for $\boldsymbol{M}$ and C is defined by

$$\nabla \cdot \boldsymbol{M} = 0, \tag{8.24}$$

$$-\rho(\boldsymbol{K}\cdot\nabla\boldsymbol{L}+\boldsymbol{L}\cdot\nabla\boldsymbol{K}) = -\nabla C+\mu\nabla^2\boldsymbol{M} \tag{8.25}$$

with the same boundary conditions (8.6) or (8.20). Its solution can be averaged to give

$$\langle u^{(2)}\rangle = -\langle M_x\rangle\left(\frac{\partial p^{(0)}}{\partial X}\right)^3, \quad \langle w^{(2)}\rangle = 0. \tag{8.26}$$

Combining the first three orders $O(\delta^0)$, $O(\delta^1)$, $O(\delta^2)$ and writing

$$K = \langle K_x\rangle, \quad \beta = \langle M_x\rangle \tag{8.27}$$

we get

$$-K\frac{\partial p}{\partial x_2} = \langle u\rangle - \frac{\delta^2}{K^3}\beta\langle u\rangle^3. \tag{8.28}$$

Thus the nonlinear extension of Darcy's law is also in the form of (6.4), despite the anisotropy here. As in §7 it can also be shown that $\beta \leqslant 0$.

Again the inertia correction to the linear law of Darcy is relatively insignificant for small flow rates. This result is not inconsistent with the numerical study of Payatakes *et al.* (1973) who calculated the friction factor for flows through a corrugated tube by solving the full Navier–Stokes equations by finite differences. The tube radius $r(x)$ varies periodically in x from $r_{\max}$, where there is sharp corner (a discontinuity in $r'(x)$), to $r_{\min}$. Within a wavelength λ between two successive $r_{\max}$, $r(x)$ is a parabola. By plotting the friction coefficient

$$f = -\frac{\Delta p}{2\lambda}\frac{D}{\rho u} \tag{8.29}$$

against the Reynolds number

$$R = \frac{\rho u D}{\mu} \tag{8.30}$$

they find that

$$\log_{10}(fR) = F\left(\frac{\Delta r}{\lambda}\right), \quad \text{where} \quad \Delta r = r_{\max} - r_{\min} \tag{8.31}$$

for $0.1 < R < 10$ and $\Delta r/\lambda = 0, 0.1, 0.3, 0.5$. Departure from (8.31) is still small for $10 < R < 100$. These computed results are in reasonable agreement with the experiments by Batra (1969) for corrugated tubes that are only roughly periodic with some irregularities. Since

$$u = -\frac{\Delta p}{2\lambda}\frac{D^2}{\mu}\frac{1}{fR} = -K\frac{\Delta p}{2\lambda}, \tag{8.32}$$

the implied coefficient $K \propto D^2/fR$ is also independent of R for R less than 10. Thus for this geometry, the linear Darcy's law is again valid up to rather high Reynolds numbers when local inertia is decidedly important. This implies that the coefficient b in (6.4) should be zero or close to zero. To see whether the same is true for other corrugated tubes would be very interesting.

9. Concluding remarks

We have applied the theory of homogenization to deduce, without a closure hypothesis, the extension of Darcy's law in a rigid porous medium when the fluid inertia is small but finite. By assuming that the medium has a periodic structure on the scale of the pores, general relations between pressure and the velocity on the macroscale are derived for anisotropic inhomogeneous media. All the coefficients in the phenomenological relations are defined by boundary-value problems of Stokes flow in a unit cell on the pore scale. These boundary-value problems can be solved numerically, although the task is by no means simple as is evident in the work by Zick & Homsy (1982) who have completed calculations of the coefficient K for a medium made up of packed spheres of equal radius.

For a medium that is homogeneous and isotropic on the macroscale, the inertia effect gives rise to a correction term cubic in the averaged velocity, hence the modification to the linear Darcy's law is very minor, as is consistent with all known experiments. This is also the case for the anisotropic one-dimensional model of parallel corrugated tubes. Although expected physically, we have also proven mathematically that the coefficient of the cubic term is of such a sign as to demand a higher pressure gradient to maintain the same seepage velocity.

In view of the robustness of Darcy's law for $R \leqslant O(1)$, as evidenced by experiments for spheres and by the numerically exact solution for a corrugated tube, it is likely that the coefficient b in (1.1) is numerically small or zero for many geometries. Explicit numerical solutions of the requisite cell problems and further experiments are needed to further understand why inertia can be more important on the microscale than on the macroscale. It is also worth facing the challenge of solving the full three-dimensional Navier–Stokes equations for a unit cell in order to predict substantial modifications of Darcy's law.

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